Answer model Exam Lie groups in Physics of Nov 8, 2017

1a) $U \in SU(2)$: $U^\dagger = U^{-1} \wedge \det U = 1$

50% $\left\{ \; U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \alpha, \beta, \gamma, \delta \in \mathbb{C} \right\} \Rightarrow U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$

with $|\alpha|^2 + |\beta|^2 = 1$

25% { parameter space: $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2 \Rightarrow \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$
so as a real parameter space it is S^3

25% { S^3 is compact & connected, hence so is $SU(2)$

1b) all $\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \in SU(2)$ ~~form~~ form a $U(1)$ subgp of $SU(2)$

$U(1)$ is Abelian, hence all its irreps are 1D, so 2D is reducible

1c) $\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} e^{i\phi}\alpha & e^{i\phi}\beta \\ -e^{-i\phi}\beta^* & e^{-i\phi}\alpha^* \end{pmatrix}$

$\cancel{\begin{pmatrix} \gamma & \delta \\ -\delta^* & \gamma^* \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} = \begin{pmatrix} \gamma e^{i\phi} & \\ & \end{pmatrix}}$ equate 11 $\Rightarrow \phi = \chi \wedge \phi = -\chi$ ↯

$\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} e^{i\chi} & 0 \\ 0 & e^{-i\chi} \end{pmatrix} = \begin{pmatrix} \alpha e^{i\chi} & \beta e^{-i\chi} \\ -\beta^* e^{i\chi} & \alpha^* e^{-i\chi} \end{pmatrix}$

$U(1)$ not invariant subgp (like $SO(2)$ is not) of $SU(2)$
hence $SU(2)/U(1)$ not a factor gp.

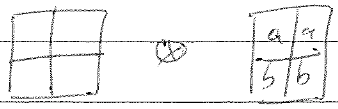
1d) $Z(SU(2))$ = all those elt's that commute with all other elt's
since defining rep is irrep, Schur's lemma says that
$D(U) = \lambda \mathbb{1} \quad \forall U \in Z(SU(2))$
For $D(U)$ to be in $SU(2)$, $|\lambda|^2 = 1 \wedge \lambda^2 = 1 \Rightarrow \lambda = \pm$
$\Rightarrow Z(SU(2)) = \mathbb{Z}_2 \quad \Leftarrow$ center is invariant subgp.
hence $SU(2)/\mathbb{Z}_2$ is a factor gp.
($SO(3)$ in fact)

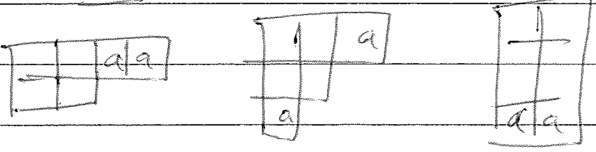
3a) $(M^{01})^\alpha{}_\beta = i(g^{0\alpha} g^1{}_\beta - g^{1\alpha} g^0{}_\beta) = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ using $g^{00} = 1$, $g^{11} = -1$, $g^\alpha{}_\beta = \delta^\alpha_\beta$

$(M^{23})^\alpha{}_\beta = i(g^{2\alpha} g^3{}_\beta - g^{3\alpha} g^2{}_\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$

Answer model (c'd)

6-11-17.2

2a) 

Step 1: 

Step 2:

bbaa not allowed

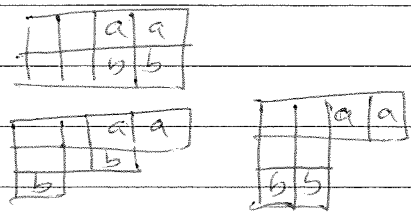
baba "

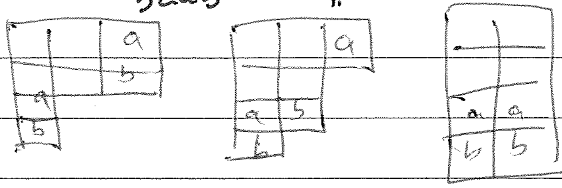
abab allowed

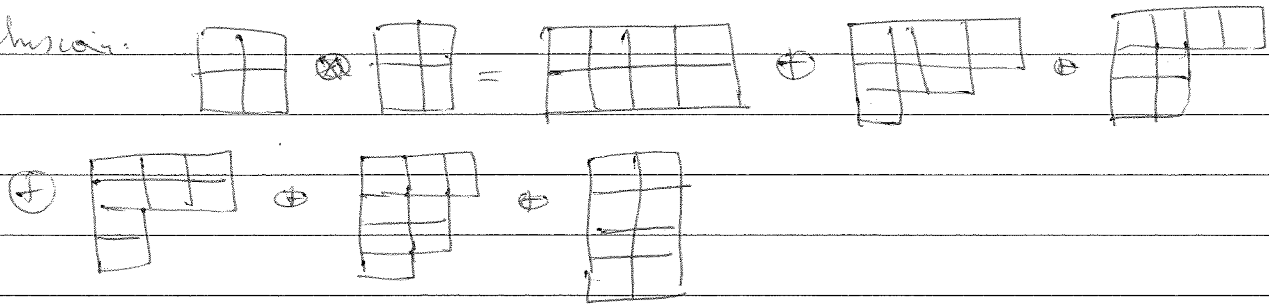
aabb "

abba not allowed

baab "



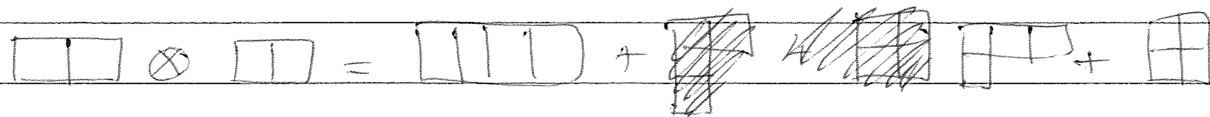


conclusion: 

2b) $(n=2: \ 1\times 1 = 1 + 0 \ldots)$

$n=3 \quad 6^*\otimes 6^* = 15_1^* + 15_2^* + 6 + 0 + 0 + 0$

$n=4 \quad 20 \otimes 20 = 105 + 175 + 84 + 20 + 15 + 1$

2c) 

$n=3 \quad 6 \times 6 = 15_1 + 15_2 + 6^*$

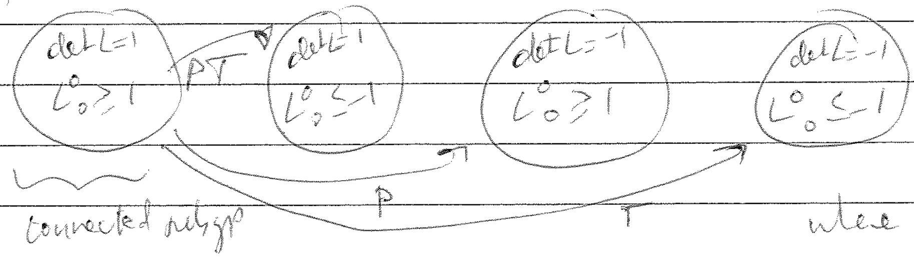
3b) $\exp(-i\chi M^{01}) = \exp\left(-i\chi\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} \exp\left(-i\chi\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}\right) & \theta \\ \theta & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\end{pmatrix}$

$$\exp\left(-i\chi\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}\right) = \sum_{n=0}^{\infty} \underbrace{\frac{(-i\chi)^{2n}}{2n!}}_{\frac{\chi^{2n}}{2n!}} \underbrace{\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}^{2n}}_{\mathbb{1}} + \sum_{n=0}^{\infty} \underbrace{(-i\chi)^{2n+1}}_{i\frac{\chi^{2n+1}}{2n+1!}} \underbrace{\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}^{2n+1}}_{\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}}$$

$$= \cosh\chi \cdot \mathbb{1} - i\sinh\chi\begin{pmatrix} 0 & i \\ i & 0\end{pmatrix} = \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi\end{pmatrix} \Rightarrow \text{boost along } x \text{ direction}$$

Answer model (c'd) 6-11-17.3

4a) a Poincaré elt is given by (a translation ~~[illegible]~~ and a Lorentz transformation. The Lorentz gp consists of 4 connected components ($\det L = \pm 1 \wedge (L^0{}_0)^2 \geq 1$)



where $P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$P = T \odot L$ translations are all connected $T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$(a|\Lambda)$ so one arrives at $P_+^\uparrow, P_+^\downarrow, P_-^\uparrow, P_-^\downarrow$

4 connected components.

4b) Pauli-Lubanski $W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$

40 ~~80~~ %

$$W^0 = -\frac{1}{2}\epsilon^{0ijk} P_i M_{jk} \qquad \epsilon^{0ijk} = \epsilon_{ijk} \text{ since } \epsilon^{0123} = 1 \ \& \ \epsilon_{123} = 1$$

$$= -\frac{1}{2}\epsilon_{ijk} P_i M_{jk} = P_i \frac{1}{2}\epsilon_{ijk} M^{jk} = -P_i J^i = \vec{P}\cdot\vec{J} = |\vec{P}|\underset{\text{helicity}}{\lambda}$$

30 ~~80~~ %

$$W^1 = -\frac{1}{2}\epsilon^{1\nu\rho\sigma} P_\nu M_{\rho\sigma} = -\frac{1}{2}\epsilon^{10jk} P_0 M_{jk} - \frac{1}{2}\epsilon^{1j0k} P_j M_{0k}\cdot 2$$

$$= +\frac{1}{2}\epsilon_{1jk} P_0 M^{jk} - \epsilon^{0ijk} P_j M_{0k}$$

$$= P_0 J^1 + \epsilon_{1jk} P_j M^{0k} = P_0 J^1 + \epsilon_{1jk} P_j K^k$$

$$= P_0 J^1 - \epsilon_{1jk} P^j K^k = P_0 J^1 - (\vec{P}\times\vec{K})$$

$$= E J^1 - (\vec{P}\times\vec{K})_1$$

using $P^\mu = (P^0, \vec{P})$

$P_\mu = (P^0, -\vec{P})$ etc

~~[illegible]~~ 30%

massive particle in rest frame: $P^\mu = (E, \vec{0})$

$\Rightarrow$ $W^0 = 0$ ~~(helicity is then undetermined %.)~~

$W^1 = E J^1 = M J^1$

4c) $W_\mu P^\mu \propto \epsilon^{\mu\nu\rho\sigma} P_\mu P_\nu M_{\rho\sigma} = 0$ because ~~the~~ ϵ is antisymmetric in $\mu \leftrightarrow \nu$ and $P_\mu P_\nu$ is symmetric.